LIBERTY PAPER SET

STD. 12 : Mathematics

Full Solution

Time : 3 Hours **ASSIGNMENT PAPER 2**

PART A

1. (C) **2.** (D) **3.** (A) **4.** (B) **5.** (C) **6.** (B) **7.** (D) **8.** (C) **9.** (C) **10.** (A) **11.** (B) **12.** (B) **13.** (D) **14.** (D) **15.** (A) **16.** (A) **17.** (C) **18.** (B) **19.** (B) **20.** (C) **21.** (A) **22.** (C) **23.** (A) **24.** (D) **25.** (B) **26.** (A) **27.** (B) **28.** (C) **29.** (D) **30.** (B) **31.** (A) **32.** (C) **33.** (A) **34.** (C) **35.** (B) **36.** (C) **37.** (D) **38.** (A) **39.** (B) **40.** (D) **41.** (D) **42.** (C) **43.** (B) **44.** (A) **45.** (B) **46.** (C) **47.** (A) **48.** (D) **49.** (C) **50.** (D)

PART B

SECTION A

1.

R.H.S. = $cos^{-1}(4x^3 - 3x)$

Suppose, $x = cos\ \theta$

$\therefore\ \theta = cos^{-1} x,\ \theta \in [0, \pi]$

$\therefore$ R.H.S. $= cos^{-1}(4cos^3\theta - 3cos\ \theta)$

$= cos^{-1}(cos 3\theta)$

Here, $\frac{1}{2} \le x \le 1$

$\therefore\ cos\ \frac{\pi}{3} \ge cos\ \theta \ge cos\ 0$

($\because$ $cos\ \theta$ is decreasing function in first quadrant)

$\therefore\ 0 \le \theta \le \frac{\pi}{3}$

$\therefore\ 0 \le 3\theta \le \pi$

$3\theta \in [0, \pi]$ .... (1)

$\therefore$ R.H.S. $= cos^{-1}(cos 3\theta)$

$= 3\theta$ ($\because$ From equation (1))

$= 3\ cos^{-1} x$

= L.H.S.

2.

R.H.S. $= \frac{1}{2}\ cos^{-1}\left[\frac{1-x}{1+x}\right]$

Suppose, $x = tan^2\theta$

$tan\theta = \sqrt{x}$

$\therefore\ \theta = tan^{-1}\sqrt{x},\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$= \frac{1}{2}\ cos^{-1}\left[\frac{1-tan^2\theta}{1+tan^2\theta}\right]$

$= \frac{1}{2} \cdot cos^{-1}(cos 2\theta)$

Here, $0 \le x \le 1$

$\therefore\ tan\ 0 \le tan\ \theta \le tan\ \frac{\pi}{4}$

$\therefore\ 0 \le \theta \le \frac{\pi}{4}$

$\therefore\ 0 \le 2\theta \le \frac{\pi}{2}$

$2\theta \in \left[0, \frac{\pi}{2}\right] \subset [0, \pi]$ .... (1)

$= \frac{1}{2}(2\theta)$ ($\because$ From equation (1))

$= \theta$

$= tan^{-1}\sqrt{x}$

= L.H.S.

3.

f is continuous at $x = \frac{\pi}{2}$

$\therefore\ \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$

Now, $\lim_{x \to \frac{\pi}{2}} \left(\frac{k\ cos\ x}{\pi - 2x}\right) = 3$

$\therefore\ \lim_{x \to \frac{\pi}{2}} \frac{k\ cos\ x}{2\left(\frac{\pi}{2} - x\right)} = 3$

$$\therefore \lim_{\frac{\pi}{2}-x\to 0} \frac{k}{2}\frac{\sin\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)} = 3 \quad \left(\because x\to\frac{\pi}{2} \Rightarrow \frac{\pi}{2}-x\to 0\right)$$

$$\therefore \frac{k(1)}{2} = 3$$

$$\therefore k = 6$$

4.

$$I = \int \frac{dx}{(e^x-1)}$$

$$= \int \frac{e^x}{e^x(e^x-1)}\, dx$$

$\rightarrow$ Take, $e^x = t$

$\therefore e^x \cdot dx = dt$

$$I = \int \frac{dt}{t(t-1)}$$

$$I = \int \frac{t-(t-1)}{t(t-1)}\, dt$$

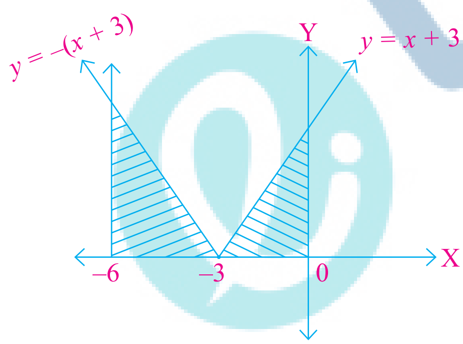
$$= \int \frac{dt}{t-1} - \int \frac{dt}{t}$$

$$= \log|t-1| - \log|t| + c$$

$$I = \log|e^x-1| - \log|e^x| + c$$

$$I = \log\left|\frac{e^x-1}{e^x}\right| + c$$

5.



$$\int_{-6}^{0} |x+3|\, dx = \int_{-6}^{-3} |x+3|\, dx + \int_{-3}^{0} |x+3|\, dx$$

$$= -\int_{-6}^{-3} (x+3)\, dx + \int_{-3}^{0} (x+3)\, dx$$

$\left(\because -6 < x < -3 \Rightarrow x+3 < 0 \Rightarrow |x+3| = -(x+3)\right)$ $\left(\because -3 < x < 0 \Rightarrow x+3 > 0 \Rightarrow |x+3| = x+3\right)$

$$= -\left[\frac{x^2}{2}+3x\right]_{-6}^{-3} + \left[\frac{x^2}{2}+3x\right]_{-3}^{0}$$

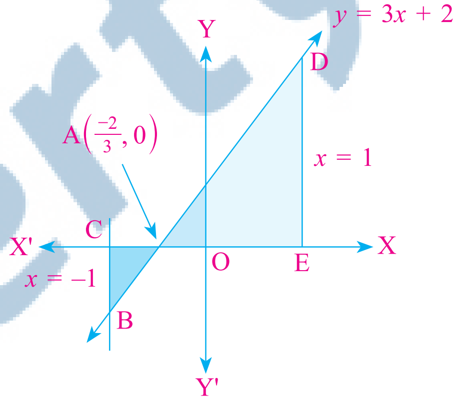
$$= -\left[\left\{\frac{9}{2}-9\right\}-\left\{\frac{36}{2}-18\right\}\right] + \left[0-\left(\frac{9}{2}-9\right)\right]$$

$$= -\frac{9}{2} + 9 - 0 - \frac{9}{2} + 9$$

$$= 9$$

6.

As shown in the fig., the line $y = 3x + 2$, meets X-axis at $\left(-\frac{2}{3}, 0\right)$ and its graph lies below X-axis for $x \in \left(-1, -\frac{2}{3}\right)$ and above X-axis for $x \in \left(-\frac{2}{3}, 1\right)$



The required area

= Area of the region ACBA + Area of the region ADEA

$$= \left|\int_{-1}^{-\frac{2}{3}} (3x+2)\, dx\right| + \int_{-\frac{2}{3}}^{1} (3x+2)\, dx$$

$$= \left|\left(\frac{3}{2}x^2+2x\right)_{-1}^{-\frac{2}{3}}\right| + \left(\frac{3}{2}x^2+2x\right)_{-\frac{2}{3}}^{1}$$

$$= \left|\left(\frac{3}{2}\left(\frac{4}{9}\right)-\frac{4}{3}\right)-\left(\frac{3}{2}(1)+2(-1)\right)\right| + \left(\frac{3}{2}(1)+2(1)\right) - \left(\frac{3}{2}\left(\frac{4}{9}\right)+2\left(-\frac{2}{3}\right)\right)$$

$$= \left|\frac{2}{3}-\frac{4}{3}-\frac{3}{2}+2\right| + \frac{3}{2} + 2 - \frac{2}{3} + \frac{4}{3}$$

$$= \left|\frac{-2}{3}-\frac{3}{2}+2\right| + \frac{3}{2} + 2 + \frac{2}{3}$$

$$= \left|\frac{-4-9+12}{6}\right| + \frac{9+12+4}{6}$$

$$= \frac{1}{6} + \frac{25}{6}$$

$$= \frac{26}{6}$$

$$= \frac{13}{3} \text{ sq. units}$$

7.

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

$$\therefore \quad \frac{dy}{dx} + \frac{2}{x}y = x \log x \qquad \ldots (1)$$

Compare given equation with $\frac{dy}{dx} + P(x)\, y = Q(x)$,

$P(x) = \frac{2}{x}$

$Q(x) = x \log x$

Integrating factor I.F. $= e^{\int P(x)\, dx}$

$= e^{\int \frac{2}{x} dx}$

$= e^{2 \log x}$

$= e^{\log x^2}$

$= x^2$

Multiply equation (1) by x^2,

$$\therefore \quad \frac{dy}{dx} x^2 + 2xy = x^3 \log x$$

$$\therefore \quad \frac{d}{dx}(y\, x^2) = \int x^3 \log x\, dx$$

$\rightarrow u = \log x\,,\ v = x^3$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\therefore \quad y \cdot x^2 = \log x \int x^3\, dx - \int\left[\frac{1}{x} \int x^3\, dx\right] dx$$

$$= \log x \cdot \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^4}{4}\, dx$$

$$\therefore \quad y \cdot x^2 = \log x \cdot \frac{x^4}{4} - \frac{x^4}{16} + c$$

$$\therefore \quad y = \log x\, \frac{x^2}{4} - \frac{x^2}{16} + cx^{-2}$$

$$\therefore \quad y = \frac{x^2}{16}(4 \log x - 1) + cx^{-2};$$

Which is required general solution of given differential equation.

8.

$|\vec{a}| = 1,\ |\vec{b}| = 1,\ |\vec{c}| = 1$

Here, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\therefore \quad |\vec{a} + \vec{b} + \vec{c}| = 0$

$\therefore \quad |\vec{a} + \vec{b} + \vec{c}|^2 = 0$

$\therefore \quad |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$

$+ 2\vec{a}\,\vec{b} + 2\vec{b}\,\vec{c} + 2\vec{c}\,\vec{a} = 0$

$\therefore \quad 1 + 1 + 1 + 2$

$(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$

$$\therefore \quad \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -\frac{3}{2}$$

9.

Line L_1 : $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$

$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$

Direction of line $\vec{b_1} = 3\hat{i} - 16\hat{j} + 7\hat{k}$

Line L_2 : $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

$\vec{r_2} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \lambda(3\hat{i} + 8\hat{j} - 5\hat{k})$

Direction of line $\vec{b_2} = 3\hat{i} + 8\hat{j} - 5\hat{k}$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$

$= 24\hat{i} + 36\hat{j} + 72\hat{k}$

$= 12(2\hat{i} + 3\hat{j} + 6\hat{k})$

$\therefore$ Direction of given line $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$A(\vec{a}) = \hat{i} + 2\hat{j} - 4\hat{k}$ line of the line

Vector equation of line,

$\therefore \quad \vec{r} = \vec{a} + \lambda\,\vec{b}\,,\ \lambda \in R$

$\therefore \quad \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

10.

Suppose, A(4, 7, 8), B(2, 3, 4),

P(–1, –2, 1), Q(1, 2, 5) are given points.

$\overrightarrow{AB} = -2\hat{i} - 4\hat{j} - 4\hat{k}$

$\overrightarrow{PQ} = 2\hat{i} + 4\hat{j} + 4\hat{k}$

Now, $\overrightarrow{AB} = \lambda\, \overrightarrow{PQ}$

$\therefore \quad (-2\hat{i} - 4\hat{j} - 4\hat{k}) = \lambda(2\hat{i} + 4\hat{j} + 4\hat{k}),\ \lambda \in R$

$\therefore \quad -2 = 2\lambda,\ -4 = 4\lambda,\ -4 = 4\lambda$

$\therefore \quad \lambda = -1,\ \lambda = -1,\ \lambda = -1$

$\therefore$ Direction ratio of $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are equal.

$\therefore$ Given both the lines are parallel.

11.

Two dice are thrown $n = 36$

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

Event A : Two numbers appearing on throwing two dice are different.

$\therefore \quad r = 30$

$$\therefore \quad P(A) = \frac{30}{36}$$

$$= \frac{5}{6}$$

Event B : The sum of numbers on the dice is 4.

B = {(1, 3), (2, 2), (3, 1)}

A ∩ B = {(1, 3), (3, 1)}

∴ $r = 2$

∴ $P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

∴ $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$

$= \frac{\frac{1}{18}}{\frac{5}{6}}$

$= \frac{1}{15}$

12.

A and B are independently try to solve problem with probability

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{3}$

Which are independent events.

∴ $P(A \cap B) = P(A) \cdot P(B)$

→ Exactly one of them solves the problem

$= P(A \cap B') + P(A' \cap B)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$= P(A) + P(B) - 2P(A)\,P(B)$

$= \frac{1}{2} + \frac{1}{3} - 2 \times \frac{1}{2} \times \frac{1}{3}$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{3}$

$= \frac{1}{2}$

SECTION B

13.

Here, $A = R - \{3\}$, $B = R - \{1\}$, $f(x) = \left(\frac{x-2}{x-3}\right)$

$\forall\, x_1, x_2 \in A, \quad f(x_1) = f(x_2)$

∴ $\frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$

∴ $(x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$

∴ $x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$

∴ $x_1 = x_2$

∴ f is one-one function.

Suppose, $y \in B = R - \{1\}$

$y = f(x)$

∴ $y = \frac{x-2}{x-3}$

∴ $y(x - 3) = x - 2$

∴ $yx - 3y = x - 2$

∴ $yx - x = 3y - 2$

∴ $x(y - 1) = 3y - 2$

∴ $x = \frac{3y-2}{y-1} \in R - \{3\}$ (Domain)

Now, $f(x) = f\left(\frac{3y-2}{y-1}\right) = \frac{\frac{3y-2}{y-1} - 2}{\frac{3y-2}{y-1} - 3}$

$= \frac{3y - 2 - 2y + 2}{3y - 2 - 3y + 3} = y$

∴ $\forall\, y \in B = R - \{1\}$ {kxu

$x = \frac{3y-2}{y-1} \in A = R - \{3\}$ such that

$f(x) = y$

∴ f is onto function.

Note : $f : R - \left\{-\frac{d}{c}\right\} \to R - \left\{\frac{a}{c}\right\}$;

$f(x) = \frac{ax+b}{cx+d}$ is always one-one and onto function.

14.

$A^T = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

∴ $P = \frac{1}{2}(A + A^T)$

$= \frac{1}{2}\left\{\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right\}$

$= \frac{1}{2}\begin{bmatrix} 6+6 & -2-2 & 2+2 \\ -2-2 & 3+3 & -1-1 \\ 2+2 & -1-1 & 3+3 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix}$

$P = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$P^T = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

∴ $P = P^T$

∴ P is symmetric matrix.

$$Q = \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore\ Q^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore\ -Q^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ Q = -Q^T$

$\therefore$ Q is skew symmetric matrix.

$$P + Q = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$= A$

15.

We have,

$|A| = 1(16 - 9) - 3(4 - 3) + 3(3 - 4) = 1 \neq 0$

Now, $A_{11} = 7$, $A_{12} = -1$, $A_{13} = -1$, $A_{21} = -3$, $A_{22} = 1$, $A_{23} = 0$, $A_{31} = -3$, $A_{32} = 0$, $A_{33} = 1$

$$\text{Therefore, } adj\ A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{Now, } A\,(adj A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= |A|\ I$

$$\text{Also, } A^{-1} = \frac{1}{|A|}\ adj\ A$$

$$= \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

16.

Differentiating with respect to t,

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}} \cdot \frac{1}{\cos^2\frac{t}{2}} \cdot \frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{-\sin^2 t + 1}{\sin t}\right)$$

$$= \frac{a\cos^2 t}{\sin t}$$

$$\frac{dy}{dt} = \frac{d}{dt}(a \sin t)$$

$$= a\cos t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}}$$

$$= \frac{\sin t}{\cos t}$$

$$\therefore\ \frac{dy}{dx} = \tan t$$

$$\therefore\ \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t)$$

$$= \sec^2 t\ \frac{dt}{dx}$$

$$= \frac{1}{\cos^2 t} \cdot \frac{1}{\frac{dx}{dt}}$$

$$= \frac{1}{\cos^2 t} \cdot \frac{1}{\frac{a\cos^2 t}{\sin t}}$$

$$= \frac{1}{a}\sec^3 t \cdot \tan t$$

17.

$$y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta,\ \theta \in \left[0, \frac{\pi}{2}\right]$$

$$\therefore\ \frac{dy}{d\theta} = \frac{(2+\cos\theta)(4\cos\theta) - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4(\cos^2\theta + \sin^2\theta) - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - 4 - 4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$\frac{dy}{d\theta} = \frac{\cos\theta\,(4-\cos\theta)}{(2+\cos\theta)^2}$$

Here, $\theta \in \left[0, \frac{\pi}{2}\right] \Rightarrow \cos\theta \geq 0$

$\Rightarrow (4 - \cos\theta) > 0$

$\Rightarrow (2 + \cos\theta)^2 > 0$

$$\Rightarrow \frac{\cos\theta\,(4-\cos\theta)}{(2+\cos\theta)^2} \geq 0$$

$$\Rightarrow \frac{dy}{d\theta} \geq 0$$

Therefore, $f(\theta)$ is increasing function in the interval of $\left[0, \frac{\pi}{2}\right]$.

18.

Let, $\vec{\beta_1} = \lambda\vec{\alpha}$ is a scalar,

i.e., $\vec{\beta_1} = 3\lambda\vec{i} - \lambda\vec{j}$

Now, $\vec{\beta_2} = \vec{\beta} - \vec{\beta_1} = (2-3\lambda)\hat{i} + (1+\lambda)\hat{j} - 3\hat{k}$

Now, since $\vec{\beta_2}$ is to be perpendicular $\vec{\alpha}$

we should have $\vec{\alpha}\cdot\vec{\beta_2} = 0$.

i.e., $3(2 - 3\lambda) - (1+\lambda) = 0$

OR $\lambda = \frac{1}{2}$.

Therefore, $\vec{\beta_1} = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}$ and $\vec{\beta_2} = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$

19.

Comparing (1) and (2) with $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ respectively

We get,

$$\vec{a_1} = \hat{i} + \hat{j},\ \vec{b_1} = 2\hat{i} - \hat{j} + \hat{k},\ \vec{a_2} = 2\hat{i} + \hat{j} - \hat{k}$$

and $\vec{b_2} = 3\hat{i} - 5\hat{j} + 2\hat{k}$

Therefore, $\vec{a_2} - \vec{a_1} = \hat{i} - \hat{k}$

and $\vec{b_1} \times \vec{b_2} = (2\hat{i} - \hat{j} + \hat{k}) \times (3\hat{i} - 5\hat{j} + 2\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= 3\hat{i} - \hat{j} + 7\hat{k}$$

So, $|\vec{b_1} \times \vec{b_2}| = \sqrt{9+1+49}$

$= \sqrt{59}$

Hence, the shortest distance between the given lines is given by

$$d = \left|\frac{(\vec{b_1}\times\vec{b_2}).(\vec{a_2}-\vec{a_1})}{|\vec{b_1}\times\vec{b_2}|}\right|$$

$$= \left|\frac{3-0+7}{\sqrt{59}}\right|$$

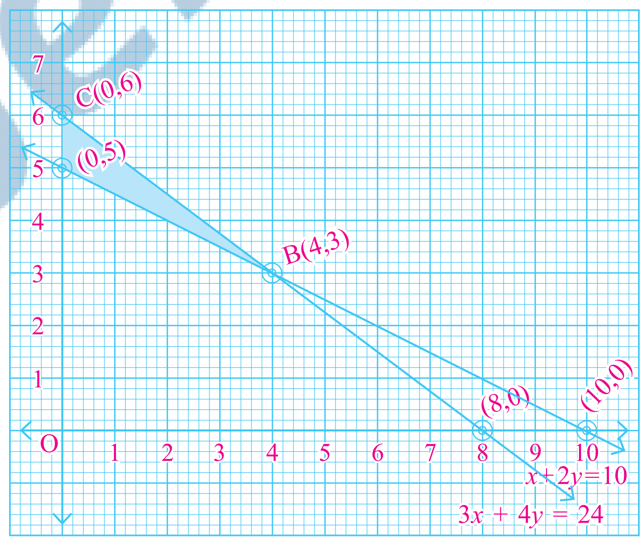
$$= \frac{10}{\sqrt{59}} \text{ unit}$$

20.

The shaded region in Fig. is the feasible region ABC determined by the system of constraints (2) to (4), which is bounded.

The coordinates of corner points A, B and C are (0,5), (4,3) and (0,6) respectively.

Now we evaluate Z = $200x + 500y$ at these points.



Corner Point	Corresponding value of Z
(0, 5)	2500
(4, 3)	2300 → Minimum
(0, 50)	3000

Hence, minimum value of Z is 2300 attained at the point (4, 3).

21.

Event E_1 : selection of ball from first bag

Event E_2 : selection of ball from second bag

Event A : second ball is red ball

$$\therefore \quad P(E_1 \mid A) = \frac{P(E_1) \cdot P(A \mid E_1)}{P(A)} \quad \text{(Bayes' rule)}$$

$$P(E_1) = \frac{1}{2} \quad ; \quad P(E_2) = \frac{1}{2}$$

$$P(A \mid E_1) = \frac{{}^4C_1}{{}^8C_1} = \frac{4}{8} = \frac{1}{2}$$

$$P(A \mid E_2) = \frac{{}^2C_1}{{}^8C_1} = \frac{2}{8} = \frac{1}{4}$$

$$\therefore \quad P(A) = P(E_1) \cdot P(A \mid E_1) + P(E_2) \cdot P(A \mid E_2)$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4}$$

$$= \frac{1}{4} + \frac{1}{8}$$

$$= \frac{3}{8}$$

Probability that the ball is drawn from the first bag which is found to be red,

$$\therefore P(E_1 \mid A) = \frac{P(A \mid E_1) \cdot P(E_1)}{P(A)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{3}{8}}$$

$$= \frac{2}{3}$$

SECTION C

22.

$A^2 = A . A$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$A^3 = A^2 . A$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

Now, L.H.S. $= A^3 - 6A^2 + 7A + 2I$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} + \begin{bmatrix} -30 & 0 & -48 \\ -12 & -24 & -30 \\ -48 & 0 & -78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30+7+2 & 0+0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0+0+0+0 & 55-78+21+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O = \text{R.H.S.}$$

$A^3 - 6A^2 + 7A + 2I = 0$

Multiplying both sides by A^{-1},

$\therefore \quad A^{-1}(A^3 - 6A^2 + 7A + 2I) = 0 \cdot A^{-1}$

$\therefore \quad A^2 - 6A + 7I + 2A^{-1} = 0$

$\therefore \quad 2A^{-1} = 6A - A^2 - 7I$

$$\therefore \quad 2A^{-1} = 6\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad 2A^{-1} = \begin{bmatrix} 6 & 0 & 12 \\ 0 & 12 & 6 \\ 12 & 0 & 18 \end{bmatrix} + \begin{bmatrix} -5 & 0 & -8 \\ -2 & -4 & -5 \\ -8 & 0 & -13 \end{bmatrix} + \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 0 & 4 \\ -2 & 1 & 1 \\ 4 & 0 & -2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}$$

23.

Let first, second and third numbers be denoted by x, y and z, respectively.

Then, according to given conditions, we have

$x + y + z = 6$

$y + 3z = 11$

$x + z = 2y$ or $x - 2y + z = 0$

This system can be written as AX = B, where,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

Hence, $|A| = 1(1 + 6) - 1(0 - 3) + 1(0 - 1) = 9 \neq 0$.

Now we find *adj* A.

$A_{11} = 1(1 + 6) = 7$,

$A_{12} = -(0 - 3) = 3$,

$A_{13} = -1$

$A_{21} = -(1 + 2) = -3$,

$A_{22} = 0,$

$A_{23} = -(-2 - 1) = 3$

$A_{31} = (3 - 1) = 2,\ A_{32} = -(3 - 0) = -3,$

$A_{33} = (1 - 0) = 1$

Hence, $adj\ A = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$

Thus, $A^{-1} = \frac{1}{|A|}\, adj(A)$

$= \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$

Since $X = A^{-1}B$

$X = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 42 - 33 + 0 \\ 18 + 0 + 0 \\ -6 + 33 + 0 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Thus, $x = 1,\ y = 2,\ z = 3.$

24.

$y = e^{a\cos^{-1}x}$

Differentiate w.r.t. x,

$$\frac{dy}{dx} = e^{a\cos^{-1}x}\, a \cdot \left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\therefore \sqrt{1-x^2}\,\frac{dy}{dx} = -\,a\, e^{a\cos^{-1}x}$$

Squaring both the sides,

$$(1 - x^2)\left(\frac{dy}{dx}\right)^2 = a^2\,[e^{a\cos^{-1}x}]^2$$

$$\therefore (1 - x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Differentiate again w.r.t. x,

$$\therefore (1 - x^2)\ 2\frac{dy}{dx}\frac{d^2y}{dx} + \left(\frac{dy}{dx}\right)^2(-2x) = a^2 \cdot 2y\ \frac{dy}{dx}$$

Now, each terms divide by $2\frac{dy}{dx} \neq 0$

$$\therefore (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$$

$$\therefore (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

25.

Rectangle □ABCD is inscribed in a r radius circle.

Length of rectangle, $AB = CD = x\ (x \neq 0,\ x > 0)$

Breadth, $BC = AD = y$

Now, $(2r)^2 = x^2 + y^2$

$\therefore\ 4r^2 = x^2 + y^2$ .... (1)

Now, Area of rectangle, A = Length × Breadth

$\therefore\ A = xy$

$\therefore\ A = x(\sqrt{4r^2 - x^2})$ ($\because$ From equation (1))

$\rightarrow$ Take, $f(x) = x^2(4r^2 - x^2)$

$f(x) = 4r^2x^2 - x^4$

$\therefore\ f'(x) = 8r^2x - 4x^3$

$\therefore\ f''(x) = 8r^2 - 12x^2$

$\rightarrow$ Now, for finding maximum area

$f'(x) = 0$

$\therefore\ 8r^2x - 4x^3 = 0$

$\therefore\ 4x(2r^2 - x^2) = 0$

$x \neq 0,\ 2r^2 - x^2 = 0$

$\therefore\ x^2 = 2r^2 \Rightarrow x = \sqrt{2}\,r$ .... (2)

Now, $f''(\sqrt{2}\,r) = 8r^2 - 12x^2$

$= 8r^2 - 12(2r^2)$

$= 8r^2 - 24r^2$

$= -16r^2 < 0$

$\therefore\ f$ has maximum value.

$\rightarrow$ From equation (1),

$4r^2 = x^2 + y^2$

$\therefore\ 4r^2 = 2r^2 + y^2$

$\therefore\ y^2 = 2r^2$

$\therefore\ y^2 = x^2$ ($\because$ From equation (2))

$\therefore\ x = y$

$\therefore$ Rectangle is square.

26.

$$I = \int_0^{\frac{\pi}{4}} \log(1 + \tan x)\, dx \quad \text{... (1)}$$

By property (6), $x = \frac{\pi}{4} - x$

$$I = \int_0^{\frac{\pi}{4}} \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1 + \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4} \cdot \tan x}\right] dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx$$

$$I = \int_0^{\frac{\pi}{4}} log\left(\frac{1+tan\,x+1-tan\,x}{1+tan\,x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} log\left(\frac{2}{1+tan\,x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} (log(2) - log(1 + tan\,x))\,dx$$

$$I = log\,2\int_0^{\frac{\pi}{4}} 1\,dx - \int_0^{\frac{\pi}{4}} log(1 + tan\,x)\,dx$$

$$I = log\,2\,[x]_0^{\frac{\pi}{4}} - I \qquad (\because \text{ From equation (1)})$$

$$2I = log\,2\left(\frac{\pi}{4} - 0\right)$$

$$\therefore I = \frac{\pi}{8}\,log\,2$$

27.

➪ Suppose, Number of Bacteria at time t is p.

Here, $\frac{dp}{dt} \propto p$

$\therefore \frac{dp}{dt} = kp$ (where, $k > 0$)

[If function is decreasing then Take $k < 0$]

$\therefore \frac{dp}{p} = k\,dt$

→ Integrate both the sides,

$\therefore \int \frac{dp}{p} = k\int 1\,dt$

$\therefore log\,|p| = kt + c$ .... (1)

→ Now, initially $t = 0$ when $p = 1{,}00{,}000$

$\therefore log\,|1{,}00{,}000| = 0 + c$

$\therefore c = log\,|1{,}00{,}000|$

→ Put the value of c in equation (1),

$\therefore log\,|p| = kt + log\,|1{,}00{,}000|$

$\therefore log\,|p| - log\,|1{,}00{,}000| = kt$

$\therefore log\left|\frac{p}{1{,}00{,}000}\right| = kt$ ... (2)

In 2 hr, Number of bacteria increases at the rate of 10%.

→ $t = 2$ hr $\Rightarrow p = 1{,}00{,}000 + (10\%$ of $1{,}00{,}000)$

$\therefore p = 1{,}00{,}000 + 1{,}00{,}000\left(\frac{10}{100}\right)$

$\therefore p = 1{,}00{,}000 + 10{,}000$

$\therefore p = 1{,}10{,}000$

$\therefore log\left|\frac{1{,}10{,}000}{1{,}00{,}000}\right| = 2k$

$\therefore 2k = log\left(\frac{11}{10}\right)$

$\therefore k = \frac{1}{2}\,log\left(\frac{11}{10}\right)$

→ Put the value of k in equation (2),

$\therefore log\left|\frac{p}{1{,}00{,}000}\right| = \frac{1}{2}\,log\left(\frac{11}{10}\right)t$

→ Now, $p = 2{,}00{,}000$ then $t = ?$

$\therefore log\left|\frac{2{,}00{,}000}{1{,}00{,}000}\right| = \frac{1}{2}\,log\left(\frac{11}{10}\right)t$

$\therefore log\,2 = \frac{1}{2}\,log\left(\frac{11}{10}\right)t$

$\therefore 2\,log\,2 = log\left(\frac{11}{10}\right)t$

$\therefore t = \frac{2\,log\,2}{log\left(\frac{11}{10}\right)}$

$\therefore t = \frac{log\,4}{log\left(\frac{11}{10}\right)}$ hr

In $\frac{log\,4}{log\left(\frac{11}{10}\right)}$ hr will the count reach 2,00,000 bacteria.